# 9.1   The Theorem of Pythagoras

In this lesson you will

- Learn about the **Pythagorean Theorem,** which states the relationship between the lengths of the legs and the length of the hypotenuse of a right triangle
- Solve a **dissection puzzle** that helps you understand the Pythagorean Theorem
- Read a **proof** of the Pythagorean Theorem
- Use the Pythagorean Theorem to **solve problems**

In a right triangle, the side opposite the right angle is called the **hypotenuse** and the other sides are called **legs.** In the figure, a and b are the lengths of the legs of a right triangle, and c is the length of the hypotenuse. There is a special relationship between the lengths of the legs and the length of the hypotenuse. This relationship is known as the Pythagorean Theorem.
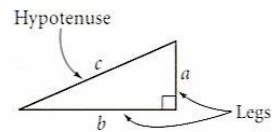
Investigation: The Three Sides of a Right Triangle

In this investigation you will solve a geometric puzzle that will help you understand the Pythagorean Theorem. You will use a **dissection,** cutting apart a figure and putting the pieces back together to form a new figure.

Construct a scalene right triangle in the middle of a piece of paper. Label the two legs a and b, and label the hypotenuse c. Construct a square on each side of the triangle so that the squares do not overlap the triangle. What is the area of each square in terms of its side length?

Now follow Steps 2–4 in your book.

After you successfully solve the puzzle in Step 4, explain the relationship among the areas of the three squares. Then use this relationship to complete the statement of the Pythagorean Theorem.

The Pythagorean Theorem In a right triangle, the sum of the squares of the lengths of the legs equals _____.   C-81

If a and b are the lengths of the two legs of a right triangle and c is the length of the hypotenuse, then a convenient way to write the Pythagorean Theorem is $a^2 + b^2 = c^2$.

A **theorem** is a conjecture that has been proved. There are over 200 known proofs of the Pythagorean Theorem. Your book gives one proof. Read the proof on page 479 of your book and make sure you can explain each step.

(continued)

Lesson 9.1 • The Theorem of Pythagoras (continued)

Page 480 of your book gives some examples that illustrate that the Pythagorean relationship, $a^2 + b^2 = c^2$, does not hold for acute or obtuse triangles.

You can use the Pythagorean Theorem to solve problems involving right triangles. Read Examples A and B in your book, and then read the examples below.

EXAMPLE A | An Olympic soccer field is a rectangle 100 meters long and 70 meters wide. How long is the diagonal of the field?

▶ **Solution** | The diagonal is the hypotenuse of a right triangle with leg lengths 70 m and 100 m. You can use the Pythagorean Theorem to find its length.
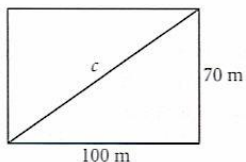
$$a^2 + b^2 = c^2 \quad \text{The Pythagorean formula.}$$

$$70^2 + 100^2 = c^2 \quad \text{Substitute the known values.}$$

$$4{,}900 + 10{,}000 = c^2 \quad \text{Square the terms.}$$

$$14{,}900 = c^2 \quad \text{Add.}$$

$$122 \approx c \quad \text{Take the positive square root of each side.}$$

The diagonal is about 122 meters long.

EXAMPLE B | What is the area of a right triangle with a leg of length 5 feet and a hypotenuse of length 13 feet?

▶ **Solution** | You can consider the two legs to be the base and the height of the triangle. The length of one leg is 5 feet. To find the length of the other leg, use the Pythagorean Theorem.
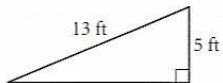
$$a^2 + b^2 = c^2 \quad \text{The Pythagorean formula.}$$

$$5^2 + b^2 = 13^2 \quad \text{Substitute.}$$

$$25 + b^2 = 169 \quad \text{Square the terms.}$$

$$b^2 = 144 \quad \text{Subtract 25 from both sides.}$$

$$b = 12 \quad \text{Take the positive square root of each side.}$$

The other leg has length 12, so the area is $\frac{1}{2} \cdot 5 \cdot 12$, or 30 square feet.

The Converse of the Pythagorean Theorem

In this lesson you will

- Experiment with **Pythagorean triples** to determine whether the converse of the Pythagorean Theorem appears to be true
- Prove the **Converse of the Pythagorean Theorem**
- Use the converse of the Pythagorean Theorem to determine whether a triangle is a right triangle

In Lesson 9.1, you learned the Pythagorean Theorem, which states that if a triangle is a right triangle, then the square of the length of its hypotenuse is equal to the sum of the squares of the lengths of the two legs. Do you think the converse is also true? In other words, if the side lengths of a triangle work in the Pythagorean equation, must the triangle be a right triangle? You'll explore this question in the investigation.

Investigation: Is the Converse True?

For this investigation you will need string, three paper clips, and two helpers. If no one is available to help you, you will need string, some pins or tacks, and a large sheet of thick cardboard.

A set of three positive integers that satisfy the Pythagorean formula is called a **Pythagorean triple.** For example, the integers 3, 4, and 5 are a Pythagorean triple because $3^2 + 4^2 = 5^2$.

Page 484 of your book lists nine examples of Pythagorean triples. Select one triple from the list, and mark off four points—A, B, C, and D—on a string to create three consecutive lengths from your set of triples. (Leave some string to the left of A and to the right of D so that you will be able to tie a knot.) For example, if you choose 5, 12, 13, you could mark your string like this:
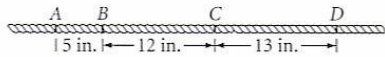

Tie the ends of the string together so that points A and D meet.

If you are working with two other people:

- Loop three paper clips onto the string.
- Three people should each pull a paper clip at point A, B, or C to stretch the string tight and form a triangle. (See the photograph in your book.)
- Measure the largest angle of the triangle. What type of triangle is formed?

If you are working by yourself:

- Pin the string to the cardboard at one of the marked points.
- Stretch the part of the string between the pinned-down point and the next marked point tight. Pin that point down. Then pull on the third marked point to stretch the string tight and form a triangle, and pin down that point.
- Measure the largest angle of the triangle. What type of triangle is formed?
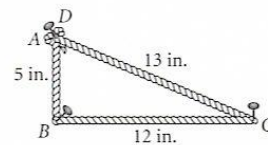
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Select at least one more triple from the list and repeat the experiment.

Your results can be summarized as a conjecture.

Converse of the Pythagorean Theorem If the lengths of the three sides of a triangle satisfy the Pythagorean equation, then the triangle is a right triangle.

C-82

On page 486 of your book, read the beginning of a proof of the Converse of the Pythagorean Theorem. Then complete the proof using the model below.

$\triangle DEF$ is a right triangle. So, by the Pythagorean Theorem, $a^2 + b^2 = x^2$.

But we were given that $a^2 + b^2 = c^2$.

Therefore, by substitution, $x^2 =$ _____.

Take the square root of each side to get _____ = _____.

We now know that $\triangle ABC \cong \triangle DEF$ by _____, so $\angle C \cong \angle F$ by _____.
Thus, $m\angle C = m\angle F = 90°$, so $\triangle ABC$ is a right triangle.

EXAMPLE

Les wanted to build a rectangular pen for his guinea pig. When he finished, he measured the bottom of the pen. He found that one side was 54 inches long, the adjacent side was 30 inches long, and one diagonal was 63 inches long. Is the pen really rectangular?

▶ **Solution**

If the pen is rectangular, then two adjacent sides and a diagonal will form a right triangle. To see if this is the case, check whether the measurements form a Pythagorean triple.
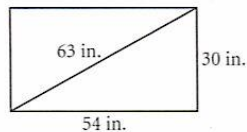
$30^2 + 54^2 = 900 + 2916 = 3816$ and $63^2 = 3969$

Because $30^2 + 54^2 \neq 63^2$, the measurements are not a Pythagorean triple, so the triangle is not a right triangle. Therefore, the pen is not rectangular.

9.3 Two Special Right Triangles

In this lesson you will

- Discover a shortcut for finding an unknown side length in an **isosceles right triangle** (also called a **45°-45°-90° triangle**)
- Discover a shortcut for finding an unknown side length in a **30°-60°-90° triangle**

An isosceles right triangle is sometimes called a 45°-45°-90° triangle because of its angle measures. Note that an isosceles right triangle is half of a square.

In the next investigation you will discover the relationship among the side lengths of an isosceles right triangle.
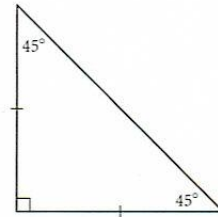
Investigation 1: Isosceles Right Triangles

The isosceles right triangle at right has legs of length l and a hypotenuse of length h. If you know the value of l, you can use the Pythagorean Theorem to find h. Here are two examples.

- If l is 5, then $h^2 = 5^2 + 5^2 = 50$, so $h = \sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}$.
- If l is 8, then $h^2 = 8^2 + 8^2 = 128$, so $h = \sqrt{128} = \sqrt{64 \cdot 2} = 8\sqrt{2}$.

Find the value of h for at least three more integer values of l. Simplify the square root, but leave it in radical form.

Look for a pattern in the relationship between l and h. Summarize your findings by completing this conjecture.
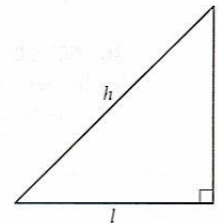
> **Isosceles Right Triangle Conjecture** In an isosceles right triangle, if the legs have length l, then the hypotenuse has length _____.

C-83

If you fold an equilateral triangle along one of its lines of symmetry, you get a 30°-60°-90° triangle, as shown at right. Triangle ABC is equilateral and $\overline{CD}$ is an angle bisector.

Prove that $\triangle ACD \cong \triangle BCD$ and then answer the questions posed on page 492 of your book. You can check your answers with the sample answers below.

1. Why must the angles in $\triangle BCD$ be 30°, 60°, and 90°?

   $m\angle B = 60°$ because $\triangle ABC$ is equilateral and therefore equiangular. By the same reasoning, $m\angle ACB = 60°$. Angle bisector $\overline{CD}$ cuts $\angle ACB$ into two congruent angles, so $m\angle ACD = 30°$ and $m\angle BCD = 30°$. Using the Triangle Sum Conjecture, $m\angle CDB = 90°$.

2. How does BD compare to AB? How does BD compare to BC?

   By the Vertex Angle Bisector Conjecture, $\overline{CD}$ is a median to $\overline{AB}$, so $AD = BD$ and $BD = \frac{1}{2}AB$. Because $\triangle ABC$ is equilateral, $AB = BC$, so $BD = \frac{1}{2}BC$.

3. In any 30°-60°-90° triangle, how does the length of the hypotenuse compare to the length of the shorter leg?

   The length of the hypotenuse is twice the length of the shorter leg.
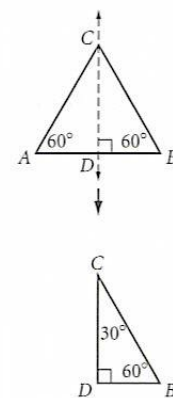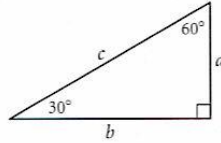
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Investigation 2: 30°-60°-90° Triangles

Below is a 30°-60°-90° triangle. If you know the length of the shorter leg, a, you can find the length of the other sides. For example, if a is 3, then the hypotenuse, c, is 6. Use the Pythagorean formula to find the length of the other leg, b.

$3^2 + b^2 = 6^2$

$9 + b^2 = 36$

$b^2 = 27$

$b = \sqrt{27} = \sqrt{9 \cdot 3} = 3\sqrt{3}$

The length of leg b is $3\sqrt{3}$ units.

Copy the table from Step 2 of Investigation 2, and repeat the procedure above to complete the table. Write the length of the longer leg in simplified radical form. Look for a pattern between the lengths of the legs of each triangle. Use your observations to complete the conjecture.
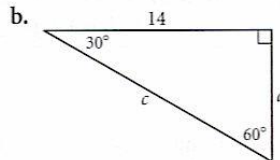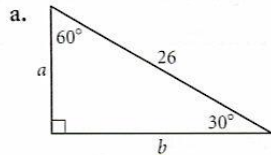
> **30°-60°-90° Triangle Conjecture** In a 30°-60°-90° triangle, if the shorter leg has length a, then the longer leg has length _____ and the hypotenuse has length $2a$.   **C-84**

Page 493 of your book gives a proof of the 30°-60°-90° Triangle Conjecture. Read the proof and make sure you understand it. Then read the example below.

EXAMPLE | Find the lettered side lengths. All lengths are in centimeters.

a.

b.

▶ **Solution** | a. The length of the shorter leg is half the length of the hypotenuse, so $a = 13$ cm. The length of the longer leg is the length of the shorter leg times $\sqrt{3}$, so $b = 13\sqrt{3}$ cm, or about 22.5 cm.

b. The length of the longer leg is the length of the shorter leg times $\sqrt{3}$, so $14 = a\sqrt{3}$ and $a = \frac{14}{\sqrt{3}}$ cm, or about 8.1 cm. The length of the hypotenuse is twice the length of the shorter leg, so $c = \frac{28}{\sqrt{3}}$ cm, or about 16.2 cm.

9.4 Story Problems

In this lesson you will

- Use the Pythagorean Theorem to **solve problems**

You can use the Pythagorean Theorem to solve many problems involving right triangles.

Read the example in your text. Notice that the problem in that example requires applying the Pythagorean Theorem twice, first to find the diagonal of the bottom of the box and then to find the diagonal of the box.

In the examples below, try to solve each problem on your own before reading the solution.

EXAMPLE A | A square has a diagonal of length 16 inches. What is the area of the square?

▸ **Solution** | The diagonal of a square divides the square into two 45°-45°-90° triangles. To find the area of the square, you need to know the leg length, l, of the triangles.
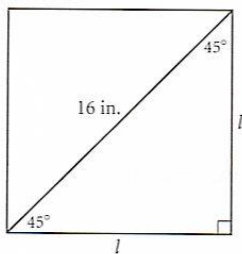
By the Isosceles Right Triangle Conjecture (or by the Pythagorean Theorem), you know that $l \cdot \sqrt{2} = 16$, so $l = \frac{16}{\sqrt{2}}$ in. Therefore,

$$\text{Area of square} = \frac{16}{\sqrt{2}} \cdot \frac{16}{\sqrt{2}}$$

$$= \frac{256}{2}$$

$$= 128$$

So, the area of the square is 128 in².

(continued)

EXAMPLE B

The Clementina High School Marching Band is practicing on the school football field. The field is 300 feet long from west to east and 160 feet wide from north to south. Len starts at the southwest corner and marches at a rate of 5 feet per second toward the southeast corner. At the same time, Jen begins marching diagonally from the northwest corner toward the southeast corner. If they want to meet at the corner at the same instant, how fast does Jen need to march?

▸ **Solution**

To start, make a sketch to illustrate the problem.
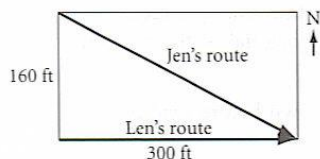
Len marches 300 feet at a rate of 5 feet per second, so it will take him $300 \div 5$, or 60 seconds, to reach the southeast corner.

For them to meet at the same time, Jen must also cover her route in 60 seconds. To find the distance Jen must march, use the Pythagorean Theorem.

$$160^2 + 300^2 = x^2$$

$$25{,}600 + 90{,}000 = x^2$$

$$115{,}600 = x^2$$

$$340 = x$$

Jen must cover 340 feet in 60 seconds, so she must march at a rate of $340 \div 60$, or about 5.7 feet per second.

9.5 Distance in Coordinate Geometry

In this lesson you will

- Learn a formula for finding the **distance between two points** on a coordinate plane
- Discover the **general equation of a circle**

On a coordinate plane, you can find the length of a segment in the *x*-direction by counting grid squares or by subtracting *x*-coordinates. Similarly, you can find the length of a segment in the *y*-direction by counting or by subtracting *y*-coordinates.

You can think of any segment that is not in the *x*- or *y*-direction as the hypotenuse of a right triangle with legs in the *x*- and *y*-directions. This allows you to use the Pythagorean Theorem to find the length of the segment.
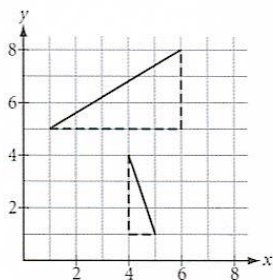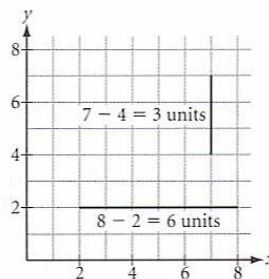
In the next investigation you'll use this idea to develop a formula for the distance between any two points on a coordinate plane.

## Investigation: The Distance Formula

Step 1 in your book shows four segments on coordinate planes. Find the length of each segment by considering it to be the hypotenuse of a right triangle. For example, the segment in part a is the hypotenuse of a right triangle with legs of lengths 2 units and 4 units, so, using the Pythagorean Theorem,

$$\text{length}^2 = 2^2 + 4^2$$

$$= 20$$

$$\text{length} = \sqrt{20} = 2\sqrt{5} \approx 4.5 \text{ units}$$

In Step 2, you must plot and connect the points and then find the distance between them. You can find the distance using the procedure you used in Step 1.

Consider the points $A(15, 34)$ and $B(42, 70)$. It wouldn't be practical to plot these points on a grid, so how can you find the distance between them?
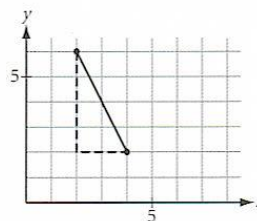
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Lesson 9.5 • Distance in Coordinate Geometry (continued)

Recall that you can find a horizontal distance by subtracting x-coordinates and a vertical distance by subtracting y-coordinates. Use this idea to complete Steps 3–5 in your book and find the distance between points $A(15, 34)$ and $B(42, 70)$.

You can generalize your findings from this investigation as a conjecture.

> **Distance Formula** The distance between points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by the formula $(AB)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ or $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.   **C-85**

Example A in your book shows how to apply the distance formula. Read Example A.

Example B shows how to use the distance formula to write the equation of a circle with center $(5, 4)$ and radius 7 units. The solution uses the fact that the circle is the set of all points (x, y) that are 7 units from the fixed point $(5, 4)$. Read Example B and then read the example below.

EXAMPLE | Write the equation for the circle at right.

▶ **Solution** | The circle has center $(0, -3)$ and radius 3 units. Let (x, y) represent any point on the circle. The distance from (x, y) to the circle's center, $(0, -3)$, is 3. Substitute this information in the distance formula.

$$(x - 0)^2 + (y - (-3))^2 = 3^2, \text{ or } x^2 + (y + 3)^2 = 3^2$$
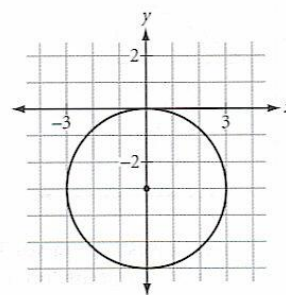
In the Mini-Investigation in Exercise 11, you'll develop the general equation of a circle.

9.6 Circles and the Pythagorean Theorem

In this lesson you will

- Use **circle conjectures** and the **Pythagorean Theorem** to solve problems

In Chapter 6, you discovered a number of properties of circles that involved right angles. Here are two of the conjectures from that chapter.

Tangent Conjecture: A tangent to a circle is perpendicular to the radius drawn to the point of tangency.

Angles Inscribed in a Semicircle Conjecture: Angles inscribed in a semicircle are right angles.

You can use these and other circle conjectures and the Pythagorean Theorem to solve some challenging problems.

Examples A and B in your book use circle conjectures, dissections, special right triangle relationships, and the Pythagorean Theorem. Read these examples and follow each step in the solutions. Below are two more examples.
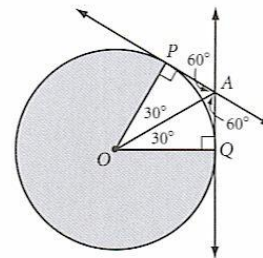
EXAMPLE A $\overleftrightarrow{AP}$ and $\overleftrightarrow{AQ}$ are tangent to circle O, and $AP = 3$ cm. Find the area of the shaded region.
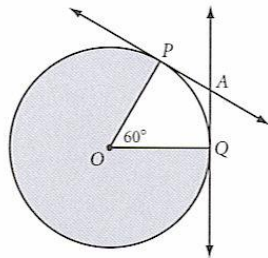
▶ **Solution**   You can draw $\overline{OA}$ to create two 30°-60°-90° triangles. (How do you know the segment bisects $\angle O$ to create two 30° angles?)

In $\triangle APO$, the shorter leg, $\overline{AP}$, has length 3 cm, so the longer leg, which is the radius of the circle, has length $3\sqrt{3}$ cm.
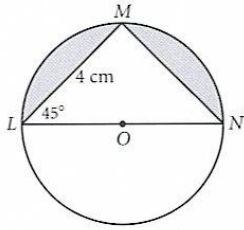

Because the radius is $3\sqrt{3}$ cm, the area of the entire circle is 27π cm². The area of the shaded region is $\frac{360 - 60}{360}$, or $\frac{5}{6}$, of the area of the circle. So, the shaded area is $\frac{5}{6}(27\pi) = \frac{45}{2}\pi$ cm², or about 70.7 cm².

(continued)

EXAMPLE B | Find the area of the shaded region.

▶ **Solution** | The area of the shaded region is the area of the semicircle minus the area of △LMN.

Because ∠LMN is inscribed in a semicircle, it is a right angle. Using the Triangle Sum Conjecture, $m\angle N = 45°$. Therefore, △LMN is an isosceles right triangle (a 45°-45°-90° triangle) with a leg of length 4 cm.

The length of the hypotenuse, which is the diameter of the circle, is $4\sqrt{2}$ cm. The radius of the circle is then $2\sqrt{2}$ cm, so the area of the entire circle is $\pi(2\sqrt{2})^2$, or 8π cm². Therefore, the area of the semicircle is 4π cm². The area of △LMN is $\frac{1}{2} \cdot 4 \cdot 4$, or 8 cm². So, the area of the shaded region is $(4\pi - 8)$ cm², or about 4.6 cm².